(a, x_0). Again divide this interval into 2 parts & continue till desired accuracy obtain otherwise it will be lie in the interval x_0 & b & perform sa[me] operation.

② Regula Falsi Method

③ General Iteration Method

④ Newton Raphson Method

General Iteration Method :-

To find a root of eqⁿ

$$f(x) = 0 \quad \text{—} ①$$

Let $x = \alpha$ be the actual root & let x_0 be initial approx. root of this eqⁿ.

We write the eqⁿ ① as

$$x = \phi(x) \quad \text{—} ② \leftarrow \text{Iteration scheme}$$

★ $\boxed{\phi'(x) < 1}$ Sufficient condⁿ

From eqⁿ ② compute the value of $\phi(x_0^*)$ that gives first approx & call it as

Second approx.

$$x_1 = \phi(x_0)$$

$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2)$$

$$\vdots$$

$$x_{n+1} = \phi(x_n)$$

That is the sequence of approx :

$\{x_k\}$, $k = 0, 1, 2,$ will be obtain

Here the func. $\phi(x)$ is called Iteration function.

Now the question arises that

① Does the sequence of approximation x_k is converge[nt]

Ans. No

(2) If it is convergent, will it always converge to (the actual root)

Ans) Yes

(3) How should we choose the initial approx. root x_0 & the iteration func. $\phi(x)$ so that the sequence x_K; $K=0,1,2,\ldots$ converges to α that is $\lim_{K\to\infty} x_K = \alpha$

Ans) The answer of this question lies in following theorem:

Theorem:- Let $x=\alpha$ with a root of the eq^n $f(x)=0$ which is equivalent to this iteration $x=\phi(x)$. Let I be an interval containing α, If $|\phi'(x)|<1|$ for every x belongs to the interval I. Then the seq. $\{x_K\}$, $K=0,1,2,\ldots$ obtained from iterative scheme.

$$\boxed{x_{n+1} = \phi(x_n)}$$ General Iteration scheme.

will converge to α provided that the initial approx. root say x_0 is choosen from the interval I.

REMARK

The cond^n $|\phi'(x)|<1$; $\forall$ $x \in I$ in the above theorem is sufficient condition for convergence.

Q.) Find the initial roots of the eq^n

$$f(x) = x^3+x^2-100=0$$

by Bolzan bisection method & then general iteration method to approx. the root correct to 3 decimal places.

$$f(x) = x^3+x^2-100=0 \quad \text{--- (1)}$$

$f(0) = -100$

$f(1) = -98$

$f(2) = -88$

$f(3) = -64$

$f(4) = -20$

$f(5) = 50$] sign changes

← initial approx. lie b/w $f(4)$ & $f(5)$

$f(4)\,f(5) < 0$

One of the roots lie b/w $f(4)$ & $f(5)$

$$x_0 = \frac{4+5}{2} = 4.5$$

We write eq^n ① as

$$x = x^3 + x^2 + x - 100 = \phi(x) \quad \text{——} ②$$

or $x = (100 - x^2)^{1/3} = \phi(x)$ —— ③ iterative f^n.

or $x = (100 - x^3)^{1/2} = \phi(x)$ —— ④

or $x = \dfrac{10}{\sqrt{1+x}} = \phi(x)$ —— ⑤

Out of these four options namely ②, ③, ④ & ⑤ we shall have to select that one which satisfies the required cond^n.

$$|\phi'(x)| < 1 \quad \forall x \in (4,5)$$

Now consider eq^n ②

$\phi(x) = x^3 + x^2 + x - 100$

$\phi'(x) = 3x^2 + 2x + 1$

$|\phi'(4)| > 1$ and $|\phi'(5)| > 1$

Product of $\phi'(4)$ and $\phi'(5) > 1$.

Thus the cond^n does not satisfied.

Now consider the eq^n ③

$\phi(x) = (100 - x^2)^{1/3}$

$\phi'(x) = \left(\frac{1}{3}\right) \dfrac{-2x}{(100 - x^2)^{2/3}}$

$|\phi'(4)| = 0.139$ & $|\phi'(5)| = 0.18$

$0.139 < 1$ $\quad$ $0.18 < 1$

Thus the cond^n satisfies so that the iterative scheme taken as

$\phi_{n+1} = \phi(x_n) = (100 - x_n^2)^{1/3}$

$x_0 = 4.5$

$x_1 = 4.304$

$x_2 = 4.335$

$x_3 = 4.330$

$x_4 = 4.331$

$x_5 = 4.331$

Consider the iterative func.

$\phi(x) = (100 - x^3)^{1/2}$

$\log_e x = \ln x$

$\log_e x = 2.303 \log_{10}$

$$|\phi'(x)| = \left| \frac{-3x^2}{(100 - x^2)^{2/3}} \right|$$

$|\phi'(4)| > 1$ & $|\phi'(5)|$ is not defined

Thus the condn is not satisfied

finally we consider last scheme taken as

$$x_{n+1} = \phi(x_n) = \frac{10}{\sqrt{1+x_n}} \quad \text{—} (6)$$

Now consider $x_0 = 4.5$

Put $x_0 = 4.5$ in eqn (6)

$x_1 = 4.264$

$x_2 = 4.359$

$x_3 = 4.320$

$x_4 = 4.336$

$x_5 = 4.329$

$x_6 = 4.332$

$x_7 = 4.331$

$x_8 = 4.331$ same

Q. Find the roots of eqn $e^{-x} - 10x = 0$ by general iteration method ~~can~~ correct to 3 decimal places.

$f(x) = e^{-x} - 10x = 0$ —(1)

$f(0) = e^{-0} - 10(0)$

$= \frac{1}{1} - 0 = 0$ (+ve)

$f(1) = e^{-1} - 10(1)$

$= \frac{1}{e} - 10 = 0.3678 - 10$

$= -9.6322$ (−ve)

$f(0)\,f(1) < 0$

one of the roots lies b/w $f(0)$ & $f(1)$

$x_0 = 1/3$ or 0.333

$x = \phi(x)$; We can write eqⁿ ①

$e^{-x} = 10x \Rightarrow x = \frac{e^{-x}}{10}$

$\phi(x) = \frac{e^{-x}}{10}$

$\phi'(x) = -\frac{e^{-x}}{10} \Rightarrow |\phi'(x)| = \left|\frac{e^{-x}}{10}\right|$

$|\phi'(x)|_{x_0 = 1/3} = \frac{e^{-1/3}}{10} = \frac{1}{1.395 \times 10} = \frac{1}{13.95} = 0.0716$

$\phi'(x) < 1 \Rightarrow |\phi'(0)| = 0.1$ & $|\phi'(1)| = 0.036$

$|\phi(0)| < 1$ $\qquad |\phi'(1)| < 1$

75

$(106-25)^{2/3}$

$(75)^{2/3}$

$\phi_{n+1} = \phi(x_n) = \frac{e^{-x_n}}{10} = x_{n+1}$

Put $n = 0$

$\phi_1 = \phi(x_0) = \frac{e^{-x_0}}{10} = x_1$

$x_1 = \frac{1}{e^{1/3}(10)} = 0.0716$

$x_2 = \frac{1}{e^{0.0716}(10)} = \frac{1}{1.0742 \times 10} = \frac{1}{10.742} = 0.0930$

$x_3 = \frac{1}{e^{0.0930} \times 10} = \frac{1}{10.974} = 0.0911$

$x_4 = \frac{1}{e^{0.0911} \times 10} = \frac{1}{10.953} = 0.0912$

$\textcircled{A} = 10\%$ correct root to 3 decimal places is 0.091

2

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Ex. The eq^n $f(x) = 3x^3 + 4x^2 + 4x + 1$ has a root in interval $(-1, 0)$. Determine the exp. f^n $\phi(x)$ such the iteration scheme $x_{n+1} = \phi(x_n)$ with $x_0 = 0.5$, converges to the root also apply the scheme to approximate the root correct to 4 decimal places.

Sol.Y $f(x) = 0 \iff x = x + \alpha f(x) = \phi(x)$ (say)

where α is arbitrary constant to be determined such that this satisfy the convergent cond^n.

$$|\phi'(x)| = |1 + \alpha(9x^2 + 8x + 4)| < 1$$

$\because$ $(9x^2 + 8x + 4) > 0$ in this interval

This imply that α is negative (<0) also the cond^n

$$|\phi'(x)| < 1 \quad \forall \; x \in (-1, 0) \Rightarrow |\phi'(-0.5)| < 1$$

or $\left|1 + \frac{9}{4}\alpha\right| < 1 \Rightarrow -1 < 1 + \frac{9}{4}\alpha < 1 \Rightarrow \frac{-8}{9} < \alpha < 0$

Thus, the range of α

Take $\alpha = -\frac{1}{2}$

The iterative scheme becomes

$$x_{n+1} = x_n - \frac{1}{2}(3x_n^3 + 4x_n^2 + 4x_n + 1)$$

Also apply this iterative scheme

$x_0 = 0.5$

$x_1 = -0.3125$

$x_2 = -0.3370$

$x_3 = -0.3327$

$x_4 = -0.3334$

$x_5 = -0.3333$

$x_6 = -0.3333$

Newton

This is an method for

Let

Newton - Raphson Method (NR)

This is a particular form of general iteration method for finding the root of the eq^n

Let x_0 be an approximate root of the eq^n $f(x) = 0$ & let $x_1 = x_0 + h$ be a better approximate root then

$$f(x_1) = f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) \cdots = 0$$

if h is small, then ~~naturally~~ h^2 & higher terms are neglected. and we get

$$h = -\frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Repeating the process we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Similarly third approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

This is known as Newton-Raphson Method

REMARK Note that in the above discussion we have assumed that h is small and that $h = -\frac{f(x)}{f'(x)}$ if $f'(x)$ is small in the neighbourhood of root then h will be large & the process will be slow & sometimes impossible. Conversely if $f'(x)$

is large in the neighbourhood of the root will be small then the root can be app[roximated] more rapidly.

Ex: Find the root of the eq^n $f(x) = x^4 - x - 10 = 0$ by Newton - Raphson method correct to 4 decimal places.

Sol: $f(0) = -10$

$f(1) = -10$

$f(2) = 4$ } sign changes

It means that the root lie in the interval $(1, 2)$

N.R. Scheme

$$x_{n+1} = x_n - \left\{ \frac{(x_n^4 - x_n - 10)}{4x_n^3 - 1} \right\}$$

Select any initial root in the interval $(1, 2)$

$x_0 = 1.7$

$$x_1 = x_0 - \left(\frac{x_0^4 - x_0 - 10}{4x_0^3 - 1} \right)$$

$x_1 = 1.8795$

$x_2 = 1.8561$

$x_3 = 1.8556$ } same.

$x_4 = 1.8556$

Q: Apply Newton Raphson method to approximate a root correct to 4 decimal places of the eq^n

$f(x) = 2x - \log_{10} x - 7 = 0$

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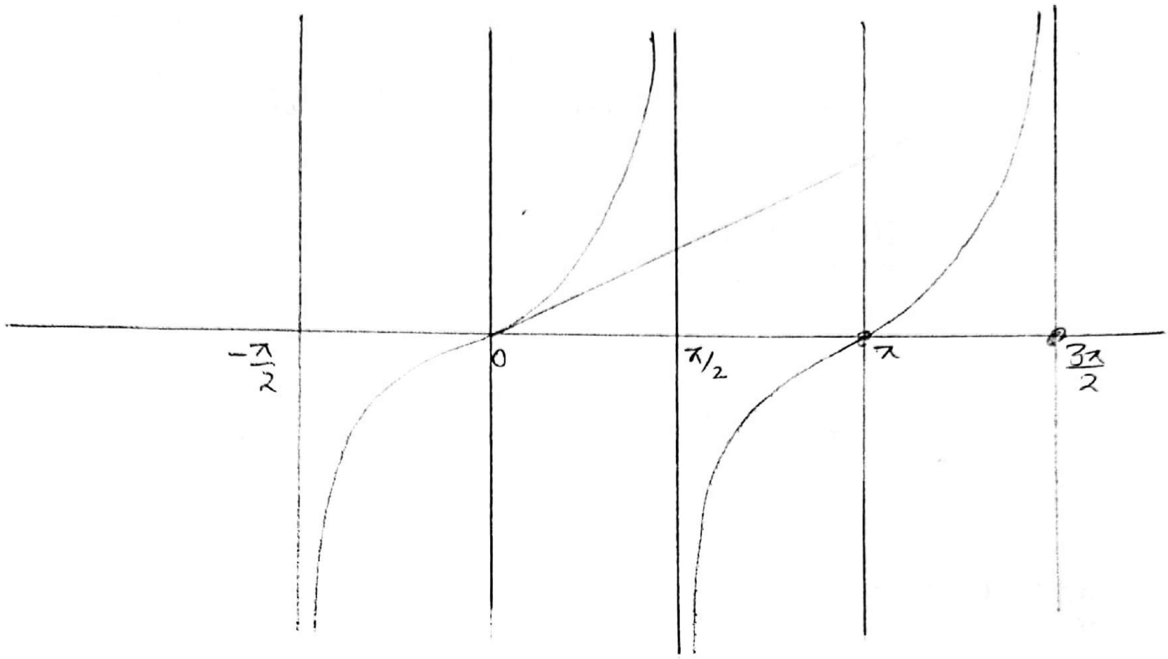
Find the smallest positive root of the eq^n, $x = \tan x$ by Newton - Raphson method correct to 5 decimal places. Obtain the initial approx. root by the graphical method.



Initial approx. of the eq^n will be any point in the neighbourhood of the intersection of $y=x$ & $y=\tan x$ i.e. a point on the left of $\frac{3\pi}{2}$ & approximately the value is 4.7 ie. regarding initial value.

By Newton - Raphson

Mathematically we choose.

$x =$	3.7	3.9	4.1	4.3	4.5
$f(x) =$	3.08	2.95	2.68	2.01	−0.14

sign changes

$\Rightarrow$ Root exists (lie) in the open interval (4.3, 4.5)

$$\boxed{x_{n+1} = x_n - \frac{x_n - \tan x_n}{1 - \sec^2 x_n}}$$

N.R. Scheme

Let $x_0 = 4.4$

$x_1 = 4.53598$

$x_2 = 4.50186$

$x_3 = 4.49735$

$x_4 = 4.49341$] same

$x_5 = 4.49341$

Q.1) Apply Newton - Raphson method to approximate the root of the following eq^n –

(i) $f(x) = x^2 + 4\sin x = 0$

(ii) ~~All~~ the +ve root of ~~four~~ eq^n $10\int_0^x e^{-x^2}dt = 0$

(iii) The positive root of the eq^n ~~f(x)~~

$$f(x) = \cos\left(\frac{\pi(x+1)}{8}\right) + 0.148x - 0.9062 = 0$$

Q.2) Using Newton - Raphson method establish the formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ to calculate $\sqrt{N}$ and hence compute $\sqrt{11}$.

Q.3) Using N-R method derive the formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ to calculate the $\sqrt{N}$ & hence find the $\sqrt{5}$ correct to 4 decimal places.

Sol.) If $x = \sqrt{N}$

Squaring on both sides

$\Rightarrow x^2 - N = 0 = f(x)$ (say)

$f'(x) = 2x$

Then by N-R formula x_n is the n^{th} iterate

then $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

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$$= \frac{x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Take $\boxed{N = 5}$, $\boxed{x_0 = 2}$.

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right)$$

$$x_1 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25$$

$$x_2 = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.2361$$

$$x_3 = \frac{1}{2}\left(2.2361 + \frac{5}{2.2361}\right) = 2.2361$$

same

Q. Derive a formula find the cube root of $\sqrt[3]{N}$ and hence find $\sqrt[3]{12}$

$N=12$ $x_0=2$.

Sol: If $x = \sqrt[3]{N}$

$\Rightarrow x^3 = N$

$\Rightarrow x^3 - N = 0 = f(x)$ (say)

$f'(x) = 3x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2}$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$$

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Take $N = 12$, choose $x_0 = 2$

$$x_1 = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$$

$x_1 = 2.3333$

$x_2 = 2.2908$

$x_3 = 2.2894$

$x_4 =$

Convergence of the Scheme

Order of convergence

Let $x = \alpha$ be ~~a~~ the actual root of the eqn $f(x)=0$ and let $\{x_k\}$, $K=0,1,2,\dots$ be a sequence of approximate roots of obtain by some iterative method.

Now,

$|x_i - \alpha| = \varepsilon_i$ (error in the i_{th} iteration $i=0,1,2,\dots$

If $\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n^p} = k$ ($\neq$, finite)

p is called the order of convergence of that iterative scheme.

Scheme for testing the order of convergence -

General Iteration Scheme

Let $x_{n+1} = \phi(x_n)$ —— (I) be an iterative scheme for finding the root of the eq^n $f(x)=0 \Leftrightarrow x = \phi(x)$ and let $x=\alpha$ be the exact root of $f(x)=0$

$$x_{n+1} = \phi(x_n)$$
$$= \phi(\alpha + x_n - \alpha)$$
(Taylor's series exp) $$= \phi(\alpha) + (x_n - \alpha)\phi'(\alpha) + \frac{(x_n-\alpha)^2}{2!}\phi''(\alpha) + \frac{(x_n-\alpha)^3}{3!}\phi'''(\alpha) \cdots$$

or $$|x_{n+1} - \alpha| = \left|(x_n-\alpha)\phi'(\alpha) + \frac{(x_n-\alpha)^2}{2!}\phi''(\alpha) + \frac{(x_n-\alpha)^3}{3!}\phi''' \cdots\right|$$

or $$|x_{n+1} - \alpha| \leq |x_n - \alpha||\phi'(\alpha)| + \frac{|(x_n-\alpha)^2|}{2!}|\phi''(\alpha)| + \frac{|(x_n-\alpha)^2|}{3!}|\phi'''(\alpha)| + \cdots$$

or $$\varepsilon_{n+1} \leq \varepsilon_n |\phi'(\alpha)| + \frac{\varepsilon_n^2}{2!}|\phi''(\alpha)| + \frac{\varepsilon_n^3}{3!}|\phi'''(\alpha)|$$

* If $\phi'(\alpha)$ is different from 0 then the order of convergence of the iterative scheme will be 1.
* If $\phi'(\alpha)$ is 0 but $\phi''(\alpha)$ is different from 0 then the order of convergance of the iterative scheme will be 2 and so on.

Q. Determine the order of of convergence of the following iterative scheme for the limit $\sqrt{a}$

(i) $x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right)$

(ii) $x_{n+1} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right)$

(iii) $\quad x_{n+1} = \frac{1}{8} x_n \left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right)$

Sol.) (i) $\quad x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right)$

Put $x_n = x_{n+1} = x$ in the iterative scheme.

we get, $\quad x = \frac{1}{2} x\left(1 + \frac{a}{x^2}\right) = \frac{1}{2}\left(x + \frac{a}{x}\right) = \phi(x)$ (say)

Here $\alpha = \sqrt{a}$

$$\phi(\sqrt{a}) = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) = \sqrt{a}$$

i.e. the condⁿ $\phi(\alpha)$ is equal to α is satisfied.

Find out the 1st derivative

$$\phi'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$$

$$\phi'(\sqrt{a}) = \frac{1}{2}\left(1 - \frac{a}{a}\right) = 0$$

$$\phi''(x) = \frac{\cancel{2}}{\cancel{2}} \frac{a}{x^3} \Rightarrow \phi''(\sqrt{a}) = \frac{a}{\sqrt{a}\cdot a} = \frac{1}{\sqrt{a}} \neq 0$$

$\therefore$ The order of convergence of iterative scheme (i) is 2 here.

(ii) $\quad x_{n+1} = \frac{1}{2} x_n\left(3 - \frac{x_n^2}{a}\right)$

Put $x_n = x_{n+1} = x$

we get, $\quad x = \frac{1}{2} x \left(3 - \frac{x^2}{a}\right)$

$$= \frac{1}{2}\left(3x - \frac{x^3}{a}\right) = \phi(x) \text{ (say)}$$

$\because \alpha = \sqrt{a}$

$$\phi(\sqrt{a}) = \frac{1}{2}\left(3\sqrt{a} - \frac{a\cdot\sqrt{a}}{a}\right) = \frac{1}{2}\cdot 2\sqrt{a}$$

$$\boxed{\phi(\sqrt{a}) = \sqrt{a}}$$

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$\because$ the condⁿ $\phi(\alpha) = \alpha$ is satisfied here.

$$\phi'(x) = \frac{1}{2}\left(3 - \frac{3x^2}{a}\right)$$

$$\phi'(\sqrt{a}) = \frac{1}{2}\left(3 - \frac{3\cancel{a}}{\cancel{a}}\right) = 0$$

$$\phi''(x) = \frac{1}{2} \times \left(-\frac{6x}{a}\right) = \frac{-3x}{a}$$

$$\phi''(\sqrt{a}) = \frac{-3\sqrt{a}}{a} = -\frac{3}{\sqrt{a}} \neq 0$$

Again here the order of convergence of iterative scheme (ii) is 2.

(iii) $$x_{n+1} = \frac{1}{8} x_n \left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right)$$

Put $x_n = x_{n+1} = x$

We get, $$x = \frac{1}{8} x \left(6 + \frac{3a}{x^2} - \frac{x^2}{a}\right)$$

$$= \frac{1}{8}\left(6x + \frac{3a}{x} - \frac{x^3}{a}\right) = \phi(x) \text{ (say)}$$

$$\alpha = \sqrt{a}$$

$$\phi(\sqrt{a}) = \frac{1}{8}\left(6\sqrt{a} + \frac{3a}{\sqrt{a}} - \frac{\cancel{a}\sqrt{a}}{\cancel{a}}\right)$$

$$\phi(\sqrt{a}) = \frac{1}{8} \times 8\sqrt{a} = \sqrt{a}$$

$\Rightarrow$ Condⁿ is satisfied here

$$\phi'(x) = \frac{1}{8}\left(6 - \frac{3a}{x^2} - \frac{3x^2}{a}\right)$$

$$\phi'(\sqrt{a}) = \frac{1}{8}\left(6 - \frac{3\cancel{a}}{\cancel{a}} - \frac{3\cancel{a}}{\cancel{a}}\right) = 0 \checkmark$$

$$\phi''(x) = \frac{1}{8}\left(0 + \frac{6a}{x^3} - \frac{6x}{a}\right) = \frac{1}{8}\left(\frac{6a}{x^3} - \frac{6x}{a}\right)$$

$$\phi''(\sqrt{a}) = \frac{1}{8}\left(\frac{6\cancel{a}}{\cancel{a}\sqrt{a}} - \frac{6\sqrt{a}}{a}\right)$$

$$\phi''(\sqrt{a}) = \frac{1}{8}\left(\frac{6}{\sqrt{a}} - \frac{6}{\sqrt{a}}\right) = 0$$

$$\phi'''(x) = \frac{1}{8}\left(-\frac{18a}{x^4} - \frac{6}{a}\right)$$

$$\phi'''(\sqrt{a}) = \frac{1}{8}\left(-\frac{18a}{a^2} - \frac{6}{a}\right)$$

$$= \frac{1}{8}\left(\frac{-24}{a}\right) = \frac{-3}{a} \neq 0$$

Here, $\phi'(\sqrt{a})$ & $\phi''(\sqrt{a})$ both are zero but $\phi'''(\sqrt{a}) \neq 0$ hence the order of convergence of the iterative scheme is 3.

Q. Determine the order of convergence of the N-R method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ... ①

Sol: Let α be the exact root of the eq^n $f(x)=0$ so that $f(\alpha)=0$, then put $\boxed{x = x_n = x_{n+1}}$ in ①

$$x = x - \frac{f(x)}{f'(x)} = \phi(x) \text{ (say)} \quad \text{—②}$$

Put $x = \alpha$

$$\alpha = \alpha - \frac{\overbrace{f(\alpha)}^{=0}}{f'(\alpha)} = \phi(\alpha) \text{ — eq}$$

We get, $\phi(\alpha) = \alpha$

Now differentiate eq^n ②

$$\phi'(x) = 1 - \frac{(f'(x))^2 - f(x)\, f''(x)}{[f'(x)]^2}$$

$$\phi'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

$\boxed{\phi'(\alpha) = 0}$ $\qquad \because f(\alpha) = 0$

$$\phi''(\alpha) = \frac{[f'(x)]^2\,[f(x)\,f''(x) + f'(x)f''(x)] - 2f(x)\,f'(x)\,[f''(x)]}{[f'(x)]^2}$$

$$\phi''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} \neq 0$$

First derivative $\phi'(\alpha) = 0$ but $\phi''(\alpha) \neq 0$ hence the order of convergence of the given iterative scheme which N-R scheme is 2 here.

Q. The eq^n $x^2 + ax + b = 0$ has 2 real roots α and β. Show that the iterative scheme:

(i) $x_{n+1} = -\left(\frac{ax_n + b}{x_n}\right)$ is convergent near to α if $\boxed{|\alpha| > |\beta|}$

ii) $x_{n+1} = -\frac{b}{x_n + a}$ is convergent near to α if $|\alpha| < |\beta|$

(iii) $x_{n+1} = -\frac{x_n^2 + b}{a}$ is convergent near to α if $2|\alpha| < |\alpha$

Sol^n ~~If~~ ~~α~~ We know that if α and β are real roots of eq^n $x^2 + ax + b = 0$, then

$\alpha + \beta = -a$ & $\alpha\beta = b$

Put $x_n = x_{n+1} = x$ in the iterative scheme

We get $x = -\frac{ax + b}{x} = -\left(a + \frac{b}{x}\right) = \phi(x)$ (say)

Also we know that for the convergence near to α

$$|\phi'(x)|_{x=\alpha} < 1 \Rightarrow \left|\frac{b}{x^2}\right|_{x=\alpha} < 1 \Rightarrow \left|\frac{\cancel{\alpha}\beta}{\alpha^{\cancel{2}}}\right| < 1$$

$\Rightarrow \left|\frac{\beta}{\alpha}\right| < 1$

$\Rightarrow \boxed{|\beta| < |\alpha|}$

(iii)

ii) We know that if α & β are real roots of the eqn $x^2+ax+b=0$, then

$$\alpha+\beta = -a, \qquad \alpha\beta = b$$

Put $x_{n+1} = x_n = x$

We get $x = \frac{-b}{x+a} = \frac{-1}{\left(\frac{x}{b}+\frac{a}{b}\right)} = \phi(x)$ (say)

$$\phi'(x) = \frac{\left(\frac{x}{b}+\frac{a}{b}\right)(0) + 1\left(\frac{1}{b}\right)}{\left(\frac{x}{b}+\frac{a}{b}\right)^2}$$

$$= \frac{\frac{1}{b}}{\left(\frac{x+a}{b}\right)^2} = \frac{b}{(x+a)^2}$$

$x^2+ax+b=0$

$\alpha+\beta=-a$

$\alpha\beta=b$

$|\phi'(x)|_{x=\alpha} < 1$

$\Rightarrow \cancel{\left|\frac{\frac{1}{\alpha\beta}}{\frac{\alpha-(\alpha+\beta)}{\alpha\beta}}\right| < 1}$

$\Rightarrow \left|\frac{\alpha\beta}{(\cancel{\alpha}-\cancel{\alpha}-\beta)^2}\right| < 1$

$\Rightarrow \left|\frac{\alpha\cancel{\beta}}{\beta^{\cancel{2}}}\right| < 1$

$\Rightarrow |\alpha| < |\beta|$

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(iii) We know that if α & β are real roots of the eq^n $x^2+ax+b=0$, then

$$\alpha+\beta=-a \quad \& \quad \alpha\beta=b$$

Put $x_{n+1}=x_n=x$

We get, $x=-\dfrac{x^2+b}{a}=\phi(x)$ (say)

Also we know that for convergence near to α

$$|\phi'(x)|_{x=\alpha}<1$$

$$\phi'(x)=-\left\{\frac{a(2x)-(x^2+b)(0)}{a^2}\right\}$$

$$=-\frac{2ax}{a^2}=\frac{-2x}{a}$$

$$|\phi'(x)|_{x=\alpha}<1$$

$$\Rightarrow \left|-\frac{2x}{a}\right|_{x=\alpha}<1$$

$$\Rightarrow \left|\frac{+2\alpha}{+(\alpha+\beta)}\right|<1$$

$$|2\alpha|<|\alpha+\beta|$$

Solution of system of linear Equations

There are several method for solving the system of eq^n

1. Cramer's Rule
2. Matrix - Inversion
3. Factorisation
4. Gauss - Elimination
5. Gauss - Jordan
6. Gauss - Seidel
7. Jacobian Method

Gauss- Elimination Method:-

This is elementary elimination method and it reduces the system to an upper triangular system which can be solved by back substitution method. For the sake of clearity & simplicity we shall demonstrate a method by considering system of 3 eq^n.

Consider a system of eq^n:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The augumented matrix

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Step-I: Elimination of x from the 2nd 3rd eqⁿ, multiply ① eqⁿ by $(-a_2/a_1)$ and adding to ② eqⁿ. Similarly multiplying ① eqⁿ by $(-a_3/a_1)$ and adding to ③ eqⁿ (Assuming a_1 is different from 0)

Here the ① eqn will be called "pivotal eqⁿ" and the leading coefficient a_1 is called the first pivot and the multipliers $(-a_2/a_1)$ & $(-a_3/a_1)$ are called the first stage multiplier.

At the end of first stage the augmented matrix becomes

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right)$$

$$b_2' = b_2 - (-a_2/a_1)\,b_1$$

$$b_3' = b_3 - (-a_3/a_1)\,b_1$$

$$c_2' = c_2 - (-a_2/a_1)\,c_1$$

$$c_3' = c_3 - (-a_3/a_1)\,c_1$$

$$d_2' = d_2 - (-a_2/a_1)\,d_1$$

$$d_3' = d_3 - (-a_3/a_1)\,d_1$$

Step-II: Elimination of y from ③ eqⁿ, now ② eqⁿ will be the pivotal eqⁿ and b_2' will be the new pivot. Multiplier will be $(-b_3'/b_2')$ and performing the operations i.e. multiplying ② eqⁿ by $(-b_3'/b_2')$ and adding the ③ eqⁿ, the system reduces to -

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2'' & c_2'' & d_2'' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right)$$

~~Where~~ Where, $c_3'' = c_3' - \left(\frac{b_3'}{b_2'}\right) c_2'$

~~and so the~~

$\boxed{\text{REMARK}}$

1. If any one of the pivot is 0 (vanish) the method will fail. In that case we rearrange the eq^n to overcome such situations.
2. In this method we select the pivotal eq^n which has numerically the largest coefficient of unknown which we want to eliminate so that the multiplier are small and the errors are not propagated. This modified form of ~~elimatio~~ elimination is called partial pivoting & this is to be carried out at every stage.

~~S.Q~~ Solve the following system of equations by Gauss-Elimi: method with partial pivoting

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$

$$8x - 3y + 2z = 20$$
$$2x + y + 4z = 12$$
$$4x + 11y - z = 33$$

Elimination of x makes eq^n (2) pivotal eq^n Pivot & multipliers are 8 and $-2/8$, $-4/8$ respective. Thus rearranging the system we have

$R_2 \to R_2 - \frac{R_1}{4}$

$12 - \frac{20}{4}$ (7)

$$\begin{array}{c} \\ -2/8 \\ -4/8 \end{array} \left(\begin{array}{ccc|c} 8 & -3 & 2 & 20 \\ 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 8 & -3 & 2 & 20 \\ 0 & 1.75 & 3.5 & 7 \\ 0 & 12.5 & -2 & 23 \end{array}\right) \longrightarrow \left(\frac{-1.75}{12.5}\right)$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 8 & -3 & 2 & 20 \\ 0 & 12.5 & -2 & 23 \\ 0 & 0 & 3.78 & 3.78 \end{array}\right)$$

By using the back substitution, $\boxed{z=1}$

$\boxed{y=2}$

$\boxed{x=3}$

Q. Solve the system of equations by Gauss-Elim^n. method by partial pivoting

$$\begin{bmatrix} 3.15 & -1.96 & 3.58 \\ 2.13 & 5.12 & -2.98 \\ 5.92 & 3.05 & 2.15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

~~Q1~~ Use Gauss Elim^n method with partial pivoting and solve the system of equations:-

$$x+y+z=7$$
$$3x+3y+4z=24$$
$$2x+y+3z=16$$

~~Q2~~ Solve with and without partial pivoting the following system of equations and observe the effect of pivoting.

$$0.0003x+3y=2.0001$$
$$x+y=1$$

$$x_n^{r+1} = \frac{1}{a_{nn}} \left(b_n - a_{n1} x_1^{r+1} - a_{n2} x_2^{r+1} \cdots - a_{nn-1} x_{n-1}^{r+1} \right)$$

Remark

1. Gauss Seidel method is convergent for any choice of initial approximation if every eq^n of the system satisfy the condition that the sum of absolute values of the coeff $\left(a_{ij}/a_{ii} \right)$ is almost equal to ~~all in~~ or in at least one eq^n less than unity.

$$\sum_{j=1}^{n} \frac{|a_{ij}|}{|a_{ii}|} \leq 1 \ \forall i$$

If the diagonal element of every row is greater or equal to the sum of the non-diagonal elements of the same row $|a_{ii}| \geq \Sigma |a_{ij}|$

where '<' should be valid in at least one eq^n.

2. The above mentioned cond^n is called diagonally dominant cond^n & it is only sufficient cond^n.

Q.4 Perform 3 iterations of {Gauss-Seidel iteration} method to the following system of equations

$$x - 8y + 3z = -4 \quad p.$$

$|-8| > |1| + |3| \Rightarrow 8 > 4$

$$2x + y + 9z = 12 \rightarrow 2$$

$$8x + 2y - 2z = 8 \longrightarrow n.$$

Rearranging the eq^n so that it becomes diagonally dominant

$$8x + 2y - 2z = 8$$
$$2x + y + 9z = 12$$
$$x - 8y + 3z = -4$$

Observe that the system is diagonally dominant

$$x^{(1)} = \frac{1}{8} \left(8 - 2y + 2z \right) = 1$$

$$y^{(1)} = \frac{1}{8}(4 + x + 3z) = 0.625$$

$$z^{(1)} = \frac{1}{9}(12 - 2x - y) = 1.0417$$

Let the initial approx^n, $x^{(0)} = y^{(0)} = z^{(0)} = 0$

$$x^{(1)} = 1$$

$$y^{(1)} = 0.625$$

$$z^{(1)} = 1.0417$$

$$x^{(2)} = \frac{1}{8}(8 - 2y^{(1)} + 2z^{(1)})$$

$$= \frac{1}{8}(8 - 1.25 + 2.0834) = 1.104$$

$$y^{(2)} = \frac{1}{8}(4 + 1 + 3.1251) = 1.0156$$

$$z^{(2)} = \frac{1}{9}(12 - 2 - 0.625) = 9.375$$

Similarly, $x^{(3)} = \frac{1}{8}(8 - 2y^{(2)} + 2z^{(2)})$

$$= \frac{1}{8}(8 - 2.0312 + 18.75) = 24.7188$$

$$y^{(3)} = \frac{1}{8}(4 + 1.104 + 28.125) = 4.1536$$

$$z^{(3)} = \frac{1}{9}(12 - 2.208 - 1.0156) = 1.097$$

Q) Solve the following system of eq^n

$$\begin{pmatrix} 0.1 & 7.0 & 0.3 \\ 0.3 & -0.2 & 10 \\ 3.0 & -0.1 & -0.2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -19.3 \\ 71.4 \\ 7.85 \end{pmatrix}$$

by gauss- elimina^n method with partial pivoting and show all steps of cal

Q) Solve the following system of linear eq^n –

$$2x + 3y - 5z = 23$$
$$3x - 4y + z = -8$$
$$5x + y - 2z = 13$$

by Gauss - siedel iterative method upto 3 iterations. Take the initial approx^n $(1.5, 2.5, 2.5)^T$

FINITE DIFFERENCE

Forward Finite Difference

Forward difference operator Δ

Let $y = f(x)$ be a f^n of independent variable x. The values of x are called arguments & correspond-ing values of y are called entries. The difference b/w 2 consecutive value of x are interval of difference.

If the interval of differencing be 'h' and the first argument be x_0

Argument x : x_0 $\quad x_0+h \quad x_0+2h \quad x_0+3h$

Entries y : $y_0 \quad y_1 \quad y_2 \quad y_3$

$y_1 - y_0$ is defined as in terms of funct^n

$$y_1 - y_0 = f(x_0+h) - f(x_0) = \Delta y_0$$

$$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$$

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Q) Solve the following system of equations by Siedel method. performing 3 iterations

$$2x-3y+5z=10$$
$$5x+y-2z=2$$
$$3x+4y-z=-2$$

Rearranging the eqⁿ so that it becomes diagonally dominant, i.e.

$$5x+y-2z=2$$
$$2x-3y+5z=10$$
$$3x+4y-z=-2$$

$$\Downarrow$$

~~After Rearranging~~

$$5x+y-2z=2$$
$$3x+4y-z=-2$$
$$2x-3y+5z=10$$

$$x = \frac{1}{5}(2-y+2z)$$

$$y = \frac{1}{4}(-2-3x+z)$$

$$z = \frac{1}{5}(10-2x+3y)$$

~~The~~ Taking the initial approxⁿ, $x^{(0)} = y^{(0)} = z^{(0)} = 1$

Then Gauss-Seidel iterative scheme -

$$x^{r+1} = \frac{1}{5}(2-y^{r}+2z^{r})$$

$$y^{r+1} = \frac{1}{4}(-2-3x^{r+1}+z^{r})$$

$$z^{r+1} = \frac{1}{5}(10-2x^{r+1}+3y^{r+1})$$

r	x^{r+1}	y^{r+1}	z^{r+1}
0	$2/5 = 0.4$	-0.8	1.36
1			
2			

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Similarly ~~#~~ we can define difference of third orde

Properties of F.D operator Δ

1.) $\Delta c = 0$, where c is constant

2.) $\Delta(cf(x)) = c \cdot \Delta f(x)$

3.) $\Delta[af(x)+bg(x)] = a\Delta f(x) + b\cdot \Delta g(x)$

4.) The n^{th} difference of n^{th} degree polynomial is constant and equal to $[(\text{coeff of } x^n) n! \cdot h^n]$ and hence the higher order differences are zero.

Q.) Prove that -

(i) $\Delta[f(x) \cdot g(x)] = f(x+h) \cdot \Delta g(x) + g(x) \cdot \Delta f(x)$

(ii) $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)\, g(x)}$

Proof -

(i) $\Delta[f(x) \cdot g(x)] = f(x+h)\, g(x+h) - f(x)\, g(x)$

$= f(x+h)\, g(x+h) - f(x+h) g(x) + f(x+h) g(x) - f(x)\, g(x)$

$= f(x+h)\,[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$

$= f(x+h) \cdot \Delta g(x) + g(x) \cdot \Delta f(x)$

(ii) $\Delta\left[\frac{f(x)}{g(x)}\right] = \left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right) = \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$

$= \frac{f(x+h)g(x) - g(x)f(x) + g(x)f(x) - f(x)\,g(x+h)}{g(x+h)g(x)}$

$= \frac{g(x)\{f(x+h) - f(x)\} - f(x)\{g(x+h) - g(x)\}}{g(x+h)\, g(x)}$

$= \frac{g(x) \cdot \Delta f(x) - f(x) \cdot \Delta g(x)}{g(x+h)\, g(x)}$

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Q. Evaluate the following, interval of difference unity. (1).

(i) $\Delta \tan^{-1} x$ (ii) $\Delta \left[\dfrac{2^x}{(x+1)!}\right]$ (iii) $\Delta \left[\dfrac{e^x}{e^x + e^{-x}}\right]$

(iv) $\Delta (e^{2x} \cdot \log 3x)$

Proof –

(i) $\Delta \tan^{-1} x = \tan^{-1}(x+1) - \tan^{-1} x$

$$= \tan^{-1}\left[\frac{(x+1)-x}{1+(x+1)x}\right]$$

$$= \tan^{-1}\left[\frac{1}{1+x(x+1)}\right]$$

(ii)

Q) Evaluate the following, the interval of diff be h.

(i) $\Delta(x^2 + \sin x)$

(ii) $\Delta(\sin 2x \cos 4x)$

(iii) $\Delta \cot a^x$

(iv) $\Delta\left(\dfrac{x^2}{\sin 2x}\right)$.

Proof –

(i) $\Delta(x^2 + \sin x)$

$$= (x+h)^2 - x^2 + \sin(x+h) - \sin x$$

$$= x^2 + h^2 + 2xh - x^2 + 2\cos\left(\frac{(x+h)+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$$

$$= h^2 + 2xh + 2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

Q) Evaluate $\Delta^n \cos(cx+d)$

Sol) $$\Delta \cos(cx+d) = \cos[c(x+h)+d] - \cos(cx+d)$$
$$= -2\sin\left(cx+d+\frac{ch}{2}\right)\sin\frac{ch}{2}$$
$$= 2\sin\frac{ch}{2}\cos\left[\frac{\pi}{2}+\left(cx+d+\frac{ch}{2}\right)\right]$$
$$\boxed{\Delta\cos(cx+d) = 2\sin\frac{ch}{2}\cos\left[cx+d+\frac{ch+\pi}{2}\right]}$$

~~By using the first f~~

Thus the first difference of $\cos(cx+d)$ is obtained by multiplying by the constant factor $2\sin\frac{ch}{2}$ and increasing the angle by $\left(\frac{ch+\pi}{2}\right)$.

$$\Delta^2 \cos(cx+d) = \Delta(\Delta\cos(cx+d))$$
$$= \Delta\left[2\sin\frac{ch}{2}.\cos\left\{cx+d+\frac{ch+\pi}{2}\right\}\right]$$
$$= 2\sin\frac{ch}{2}\cos\left(c(x+h)+d+\frac{ch+\pi}{2}\right) - 2\sin\frac{ch}{2}\cos\left\{cx+d+\frac{ch+\pi}{2}\right\}$$
$$= 2\sin\frac{ch}{2}\left[\cos\left\{c(x+h)+d+\frac{ch+\pi}{2}\right\} - \cos\left\{cx+d+\frac{ch+\pi}{2}\right\}\right]$$

$$\boxed{\cos C - \cos D = -2\sin\left(\frac{C-D}{2}\right)\sin\left(\frac{C+D}{2}\right)}$$

$$= \left(2\sin\frac{ch}{2}\right)^2\cos\left[cx+d+2\left(\frac{ch+\pi}{2}\right)\right]$$

Similarly we can find the n^{th} differency

$$\Delta^n\cos(cx+d) = \left(2\sin\frac{ch}{2}\right)^n\cos\left[cx+d+n\left(\frac{ch+\pi}{2}\right)\right]$$

$$= 2\sin\frac{ch}{2}\left[-2\sin\left\{\frac{\left(c(x+h)+d+\frac{ch+\pi}{2}\right)-\left(cx+d+\frac{ch+\pi}{2}\right)}{2}\right\}.\sin\left\{\frac{\left(c(x+h)+d+\frac{ch+\pi}{2}\right)+\left(cx+d+\frac{ch+\pi}{2}\right)}{2}\right\}\right.$$

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$$= 2\sin\frac{ch}{2}\left[-2\sin\frac{ch}{2}\cdot\sin\left\{\frac{c(x+h)+cx+2d+2\left(\frac{ch+\pi}{2}\right)}{2}\right\}\right]$$

$$=\left(2\sin\frac{ch}{2}\right)^2\left[-\sin\left\{\frac{cx+ch+cx+2d+ch+\pi}{2}\right\}\right]$$

$$=\left(2\sin\frac{ch}{2}\right)^2\left[-\sin\left(\frac{2cx+2ch+2d+\pi}{2}\right)\right]$$

$$=\left(2\sin\frac{ch}{2}\right)^2\cos\left[\frac{\pi}{2}+\frac{2cx+2d+2ch+\pi}{2}\right]$$

$$\boxed{=\left(2\sin\frac{ch}{2}\right)^2\cos\left[(cx+d)+2\left(\frac{ch+\pi}{2}\right)\right]=\Delta^2\cos(cx+d)}$$

Similarly, we can find the n^{th} difference

$$\boxed{\Delta^n\cos(cx+d)=\left(2\sin\frac{ch}{2}\right)^n\cos\left[cx+d+n\left(\frac{ch+\pi}{2}\right)\right]}$$

Backward Difference Operator ∇

The backward difference operator ∇ is defined by-

$$\nabla y_n = y_n - y_{n-1}$$

$n = 1, 2, 3, \ldots$

Put $n=1$, $\nabla y_1 = y_1 - y_0$

$$\nabla^2 y_1 = \nabla(\nabla y_1) = \nabla(y_1 - y_0)$$

$$= \nabla y_1 - \nabla y_0$$

Put $n=5$, $\nabla y_5 = y_5 - y_4$

~~The~~ Displacement or Shift operator E-

The operator E increases the value of the arguments by 1 interval.

$$E y_1 = y_2$$

$$E y_n = y_{n+1}$$

In general,

$$\boxed{E^r y_n = y_{n+r}}$$

$$E[f(x)] = f(x+h)$$

$$Ey = Ey_1$$

Properties of shift operator:-

(linear property)

1. $E\,c f(x) = c\,E f(x)$, where c is constant
2. $E[af(x) + b\,g(x)] = a\,E[f(x)] + b\,E[g(x)]$
3. $E^m[E^n f(x)] = E^{m+n}[f(x)]$
4. E and Δ are commutative

   i.e. $E\Delta f(x) = \Delta E f(x)$

Relation b/w Δ, ∇ and E

1. $\boxed{E = 1+\Delta}$

$$\Delta y_n = y_{n+1} - y_n = Ey_n - y_n = y_n(E-1)$$

$$\Rightarrow \quad \Delta = E-1 \quad \text{or} \quad E = 1+\Delta$$

In general $\boxed{E^n = (1+\Delta)^n}$

2. $\boxed{\nabla = 1-E^{-1}}$

$$\nabla y_n = y_n - y_{n-1} = y_n - E^{-1}y_n$$

$$= (1-E^{-1})\,y_n$$

$$\Rightarrow \quad \boxed{\nabla = 1-E^{-1}}$$

3. $\boxed{\nabla = \Delta E^{-1}}$

Q.) Evaluate $(\Delta + \nabla)^2 (x^2+x)$ where $h=1$

$\Rightarrow (\cancel{1} - E^{-1} + E - \cancel{1})^2 (x^2+x)$

$= (E^{\circ} - E^{-1})^2 (x^2+x)$

$= (E^2 - 2 + E^{-2})(x^2+x)$

$= E^2(x^2+x) - 2(x^2+x) + E^{-2}(x^2+x)$

$= E^2x^2 + E^2x - 2x^2 - 2x + E^{-2}x^2 + E^{-2}x$

$= (x+2)^2 + (x+2) - 2x^2 - 2x + (x-2)^2 + (x-2)$

$= \cancel{x^2} + 4 + \cancel{4x} + \cancel{x} + \cancel{2} - 2\cancel{x^2} - 2\cancel{x} + \cancel{x^2} + 4 - \cancel{4x} + \cancel{x} - \cancel{2}$

$= \cancel{2x^2} - \cancel{2x^2} + \cancel{2x} - \cancel{2x}$

Q. If D stands for the differential operator, $D = \frac{d}{dx}$.
Then prove.

$$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \cdots\right]$$

We know the, $Ef(x) = f(x+h)$

$= f(x) + hf(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

$= f(x) + hDf(x) + \frac{h^2}{2!}D^2f(x) + \frac{h^3}{3!}D^3f(x)$

$= \left[1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \cdots\right] f(x)$

$Ef(x) = e^{hD} f(x)$

$E = e^{hD}$

$hD = \log E = \log(1+\Delta)$

$D = \frac{1}{h}[$

Q.4 Prove that $\nabla y_{n+1} = h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \dots\right] y_n'$

Central Difference Operator, δ :-

The central difference operator is defined as-

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

Averaging operator, μ :-

The averaging operator, μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

Relation b/w the operator-

(i) $\delta = E^{1/2} - E^{-1/2}$

(ii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

(iii) $\boxed{\delta = \Delta E^{-1/2}}$

(iv) $\boxed{\delta = \nabla E^{1/2}}$

Proof :-

(i) $\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$

when $h=1$

$$\delta[f(x)] = \left[E^{1/2} - E^{-1/2}\right] f(x)$$

$$\delta = E^{1/2} - E^{-1/2}$$

(ii) $\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$

Taking $h=1$

$$\mu f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

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Q. Prove that :-

(i) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sin h^{-1}(\mu\delta)$

(ii) $\mu^2 = 1 + \frac{1}{4}\delta^2$

(iii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{1}{4}\delta^2}$

Proof :-

(i) consider the R.H.S.

$$\left(1+\frac{1}{4}\delta^2\right)^{1/2} = \left[1+\frac{(E^{1/2}-E^{-1/2})^2}{4}\right]^{1/2}$$

$$= \left[\frac{1}{2}(\cancel{4}+E-2+E^{-1})^{1/2}\right]$$

$$= \frac{1}{2}\left[2+E+E^{-1}\right]^{1/2} = \frac{1}{2}\left[(E^{1/2}+E^{-1/2})^2\right]^{1/2}$$

$$\left(1+\frac{1}{4}\delta^2\right)^{1/2} = \frac{1}{2}(E^{1/2}+E^{-1/2}) = \mu$$

Q. Prove that :-

(i) $$\Delta x^m - \frac{1}{2}\Delta^2 x^m + \frac{1.3}{2.4}\Delta^3 x^m - \frac{1.3.5}{2.4.6}\Delta^4 x^m + \dots = \left(x+\frac{1}{2}\right)^m - \left(x-\frac{1}{2}\right)^m$$

(ii) $$xy_1 + x^2y_2 + x^3y_3 + \dots = \left(\frac{x}{1-x}\right)y_1 + \left(\frac{x}{1-x}\right)^2\Delta y_1 + \left(\frac{x}{1-x}\right)^3\Delta^3 y_1$$

Proof :-

(i) Consider the L.H.S.

$$\Delta\left(1-\frac{1}{2}\Delta+\frac{1.3}{2.4}\Delta^2-\frac{1.3.5}{2.4.6}\Delta^3+\dots\right)x^m$$

$$= \Delta(1+\Delta)^{-1/2}x^m$$

$$= \Delta E^{-1/2}x^m$$

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$$= \Delta\left(x-\frac{1}{2}\right)^m$$

$$= \left(x+\frac{1}{2}\right)^m - \left(x-\frac{1}{2}\right)^m$$

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8
x_0	y_0								
x_1	y_1	Δy_0							
x_2	y_2	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$					
x_3	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_0$	$\Delta^5 y_0$			
x_4	y_4	Δy_3	$\Delta^2 y_2$		$\Delta^4 y_1$	$\Delta^5 y_1$	$\Delta^6 y_0$	$\Delta^7 y_0$	
x_5	y_5	Δy_4	$\Delta^2 y_3$	$\Delta^3 y_2$	$\Delta^4 y_2$		$\Delta^6 y_1$		$\Delta^8 y_0$
x_6	y_6	Δy_5	$\Delta^2 y_4$	$\Delta^3 y_3$	$\Delta^4 y_3$	$\Delta^5 y_2$	$\Delta^6 y_2$	$\Delta^7 y_1$	
x_7	y_7	Δy_6	$\Delta^2 y_5$	$\Delta^3 y_4$	$\Delta^4 y_4$	$\Delta^5 y_3$			
x_8	y_8	Δy_7	$\Delta^2 y_6$	$\Delta^3 y_5$					

Backward Difference Table :-

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6	∇^7	∇^8
x_0	y_0								
x_1	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_3$					
x_2	y_2	∇y_2	$\nabla^2 y_3$		$\nabla^4 y_4$				
x_3	y_3	∇y_3	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_5$	$\nabla^5 y_5$	$\nabla^6 y_6$		
x_4	y_4	∇y_4	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_7$	$\nabla^7 y_7$	
x_5	y_5	∇y_5	$\nabla^2 y_6$	$\nabla^3 y_6$		$\nabla^5 y_7$		$\nabla^7 y_8$	$\nabla^8 y_8$
x_6	y_6	∇y_6	$\nabla^2 y_7$	$\nabla^3 y_7$	$\nabla^4 y_7$	$\nabla^5 y_8$	$\nabla^6 y_8$		
x_7	y_7	∇y_7	$\nabla^2 y_8$	$\nabla^3 y_8$	$\nabla^4 y_8$				
x_8	y_8	∇y_8							

In this table $\nabla^k y_i$, where k is the no. of difference & y are entries with the subscript i lie along the diagonal sloping upward. i.e backward w.r.t. dir^n increasing 'x', hence these are called backward differences.

Central Differences

The central diff. operator δ is defined by the operator

$$\delta y_{1/2} = y_1 - y_0$$ (avg of subscript → $\frac{1+0}{2}$)

$$\delta y_{3/2} = y_2 - y_1$$

$$\delta y_{5/2} = y_3 - y_2$$

Similarly, $\delta y_{\frac{2n-1}{2}} = y_n - y_{n-1}$

(or) $\delta y_{n-1/2} = y_n - y_{n-1}$

Similarly the higher differences can be defined.

Central Difference Table :-

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		δ^4_2		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5_{5/2}$	
x_3	y_3		$\delta^2 y_3$		δ^4_3		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5_{7/2}$	
x_4	y_4		$\delta^2 y_4$		δ^4_4		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

We observed that odd differences has a fractional suffix and all even differences with the same subscript lies horizontally are centrally in the line with the corresponding value of y here.

Error Propagation :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3		$\Delta^3 y_{2}+\epsilon$	
x_4	y_4		$\Delta^2 y_{3}+\epsilon$		
		Δy_4			
x_5	$y_5+\epsilon$		$\Delta^2 y_{4}-2\epsilon$		
		$\Delta y_{5}-\epsilon$			
x_6	y_6		$\Delta^2 y_{5}+\epsilon$		
		Δy_6			
x_7	y_7		$\Delta^2 y_6$		
		Δy_7			
x_8	y_8		$\Delta^2 y_7$		
		Δy_8			
x_9	y_9				

Following observations can be made from the error table -

1. The error increases with the order of differences
2. The error spreads fan-wise and the error propagation is confined to the Δ region with the ~~the~~ vertex at a point where the error is committed.
3. The algebraic sum of error in any column is 0.
4. The max^m error in each column appears opposite to the entry y_5

Difference of Polynomial :-

Consider the n^{th} polynomial

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_n \quad (a_0 \neq 0)$$

First diff. of n^{th} degree polynomial $P_n(x)$ is again a polynomial of degree $(n-1)$. Similarly the second diff of $P_n(x)$ will be a polynomial of degree $(n-2)$. Thus the n^{th} diff. of $P_n(x)$ will be a constant and $(n+1)^{th}$ and higher differences will be 0.

Conversely if n^{th} diffs. of a tabulated f^n are constant & $(n+1)^{th}$, $(n+2)^{th}$ ... diff vanish, then the tabulated f^n represent a polynomial of degree n.

Q. In the following table ~~one~~ value of y is incorrect and that y is cubic polynomial. Locate the incorrect value of y and correct it.

x	y	Δ	Δ^2	Δ^3	Δ^4
0	25				
		-4			
1	21		1		
				2	$2 = 3+\varepsilon$
		-3			
			3		
2	18			6	$= 3-3\varepsilon$
		0			
3	18		9		
				0	$= 3+3\varepsilon$
		9			
4	27		9		
				4	$= 3-\varepsilon$
		18			
5	45		13		
		31		3	
6	76		16		
		47			
7	123				

To solve this ques. we should prepare the diff. tab ~~of~~ upto 3^{rd} diff. since y is a cubic polynomial.

Since y is cubic polynomial so third diff. of y ($\Delta^3 y$) constant for every x. It is suggested that each entry of this column ~~be~~ will be 3

$$\left(\because \frac{\sum \Delta^3 y}{N} = \frac{15}{5} = 3 \right)$$

Due to the fact that the errors will be conf. to a triangular region with the vertex at the point where the error is committed, the above shows that the error is corresponding to $x=3$ i.e. in y_3. In the 3rd diff. table error will be

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	25			
		-4		
1	21		1	
		-3		3
2	18		4	
		1		3
3	19		7	
		8		3
4	27		10	
		18		3
5	45		13	
		31		3
6	76		16	
		47		
7	123			

Q. Using a polynomial of third degree complete the record given below in the table.

x (year):	1998	1999	2000	2001	2002
y (export in tons):	443	384	$\overline{y_2}$	397	467
	y_0	y_1		y_3	y_4

Since y is a polynomial of degree 3, this means that third F.D ($\Delta^3 y$) is constant for all x and 4th F.D, 5th F.D ... will be zero for every x.

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$$\Delta^4 y_0 = 0 \Rightarrow (E-1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$\Rightarrow E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + 1 = 0$$

$$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + 1 = 0$$

$$\Rightarrow 467 - 4(397) + 6y_2 - 4(384) + 1 = 0$$

$$\Rightarrow y_2 = 369$$

In this way, by using the prop. of shift operator, we can find the missing term.